Critical fractional p-Laplacian problems with possibly vanishing potentials[*]

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Abstract

We obtain nontrivial solutions of a critical fractional p-Laplacian equation in the whole space and with possibly vanishing potentials. In addition to the usual difficulty of the lack of compactness associated with problems involving critical Sobolev exponents, the problem is further complicated by the absence of a direct sum decomposition suitable for applying classical linking arguments. We overcome this difficulty using a generalized linking construction based on the $\mathbb{Z}_2$-cohomological index.

1 Introduction and main results

For $p \in (1, \infty)$, $s \in (0, 1)$, and $N > sp$, the fractional p-Laplacian is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s \, u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} \, (u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N. \tag{1.1}$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian $(-\Delta)^s$ when $p = 2$. Some motivations that have led to the study of these kind of operators can be found in Caffarelli [7].

The operator $(-\Delta)_p^s$ leads naturally to the quasilinear problem

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a domain in $\mathbb{R}^N$. There is currently a rapidly growing literature on this problem when Ω is bounded with Lipschitz boundary. In particular, fractional p-eigenvalue problems have been studied in [15, 19, 23, 28], regularity theory in [6, 12, 18], existence theory in the subcritical case in [17], and the critical case in [27].

The corresponding problem in the whole space was recently considered in Lehrer et al. [22] and Torres [36]. In [36], the equation

$$(-\Delta)_p^s u + V(x) |u|^{p-2} u = f(x, u), \quad x \in \mathbb{R}^N \tag{1.2}$$

was studied when the potential $V \in C(\mathbb{R}^N)$ satisfies

$$\inf_{x \in \mathbb{R}^N} V(x) > 0, \qquad \mu\left(\left\{x \in \mathbb{R}^N : V(x) \le M\right\}\right) < +\infty \quad \forall M > 0,$$

where μ denotes the Lebesgue measure in $\mathbb{R}^N$, and a nontrivial solution was obtained when the nonlinearity f is p-superlinear and subcritical.

Equation (1.2) reduces to the well-known fractional Schrödinger equation

$$(-\Delta)^s u + V(x) u = f(x, u) \tag{1.3}$$

when $p = 2$. This equation, introduced by Laskin [20, 21], is an important model in fractional quantum mechanics and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. When $s = 1$, the Lévy dynamics becomes the Brownian dynamics and equation (1.3) reduces to the classical Schrödinger equation. The fractional Schrödinger equation has been widely investigated during the last decade and positive solutions have been obtained under various assumptions on V and f (see, e.g., [9, 10, 14, 31–33, 35, 38] and the references therein).

In the present paper we investigate existence of nontrivial solutions of the equation

$$(-\Delta)_p^s u + V(x) |u|^{p-2} u = \lambda K(x) |u|^{p-2} u + \mu L(x) |u|^{q-2} u + |u|^{p_s^*-2} u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $p_s^* = Np/(N - sp)$ is the fractional critical Sobolev exponent, $q \in (p, p_s^*)$, V, K, L are positive continuous functions, and $\lambda \in \mathbb{R}$, $\mu > 0$ are parameters. The semi-linear local case $p = 2$, $s = 1$ of this problem has been extensively studied in the literature, where the main feature is to impose conditions on V, K and L to gain some compactness (see, e.g., [1–5, 16, 24]). In the quasilinear case $p \ne 2$, in addition to the usual difficulty of the lack of compactness, this problem is further complicated by the absence of a direct sum decomposition suitable for applying the classical linking theorem of Rabinowitz [30]. We will overcome this difficulty by using a generalized linking theorem based on the $\mathbb{Z}_2$-cohomological index.

We shall assume that V, K and L are positive continuous functions on $\mathbb{R}^N$ satisfying:

(H_1) K and L are bounded;

(H_2) we have:

$$\lim_{|x|\to\infty} \frac{K(x)}{V(x)} = 0, \qquad \lim_{|x|\to\infty} \frac{L(x)}{V(x)^{[Np-(N-sp)q]/sp^2}} = 0.$$

Notice that these conditions allow possibly different behaviour of the potentials at infinity. If the potential V is bounded, then K and L need to be vanishing at infinity, while if V is unbounded, then K and L may converge to a constant. Denoting by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right)^{1/p}$$

the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ and setting

$$|u|_{p,V} = \left(\int_{\mathbb{R}^N} V(x) \, |u|^p \, dx \right)^{1/p},$$

we work in the reflexive Banach space

$$X = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty, \ |u|_{p,V} < \infty \right\}$$

with the norm given by

$$\|u\|^p = [u]_{s,p}^p + |u|_{p,V}^p.$$

Let also

$$L_K^p(\mathbb{R}^N) = \left\{ u : \int_{\mathbb{R}^N} K(x) \, |u|^p \, dx < \infty \right\}, \qquad L_L^q(\mathbb{R}^N) = \left\{ u : \int_{\mathbb{R}^N} L(x) \, |u|^q \, dx < \infty \right\}$$

be the weighted Lebesgue spaces, normed by

$$|u|_{p,K} = \left(\int_{\mathbb{R}^N} K(x) \, |u|^p \, dx \right)^{1/p}, \qquad |u|_{q,L} = \left(\int_{\mathbb{R}^N} L(x) \, |u|^q \, dx \right)^{1/q},$$

respectively. We have the following compactness result.

Proposition 1.1. *If (H_1) and (H_2) hold, then X is compactly embedded in $L_K^p(\mathbb{R}^N) \cap L_L^q(\mathbb{R}^N)$.*

A weak solution of equation (1.4) is a function $u \in X$ satisfying

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2} \, (u(x)-u(y)) \, (v(x)-v(y))}{|x-y|^{N+sp}} \, dxdy + \int_{\mathbb{R}^N} V(x) \, |u|^{p-2} \, uv \, dx$$

$$= \int_{\mathbb{R}^N} \left(\lambda K(x) \, |u|^{p-2} \, u + \mu L(x) \, |u|^{q-2} \, u + |u|^{p_s^*-2} \, u \right) v \, dx \quad \forall v \in X.$$

Weak solutions coincide with critical points of the C^1-functional

$$\Phi(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} \left(\frac{\lambda}{p} K(x) |u|^p + \frac{\mu}{q} L(x) |u|^q + \frac{1}{p_s^*} |u|^{p_s^*} \right) dx, \quad u \in X.$$

Recall that Φ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset X$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. Let

$$S_V = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^p}{|u|_{p_s^*}^p} > 0, \tag{1.5}$$

where $|\cdot|_{p_s^*}$ is the standard norm in $L^{p_s^*}(\mathbb{R}^N)$. Our existence results will be based on the following proposition.

Proposition 1.2. *Assume that*

$$c < \frac{s}{N} S_V^{N/sp}, \quad c \neq 0.$$

Then any $(\mathrm{PS})_c$ sequence has a subsequence converging weakly to a nontrivial solution of (1.4).

Let

$$S_{s,p} = \inf_{u \in L^{p_s^*}(\mathbb{R}^N) \setminus \{0\}, \, [u]_{s,p} < \infty} \frac{[u]_{s,p}^p}{|u|_{p_s^*}^p}$$

be the best constant in the fractional Sobolev inequality. Since V is positive, $S_V \geq S_{s,p}$. Equality holds if $V \in L^{N/sp}(\mathbb{R}^N)$. To see this, let φ be a minimizer for $S_{s,p}$ and let $y_j \in \mathbb{R}^N$ with $|y_j| \to \infty$. Then $u_j := \varphi(\cdot - y_j) \in X$ by the Hölder inequality and $|u_j|_{p,V} \to 0$ as easily seen by approximating V and φ by functions with compact supports. Since $[\cdot]_{s,p}$ and $|\cdot|_{p_s^*}$ are translation invariant, then

$$\frac{\|u_j\|^p}{|u_j|_{p_s^*}^p} = \frac{[\varphi]_{s,p}^p + |u_j|_{p,V}^p}{|\varphi|_{p_s^*}^p} \to S_{s,p}.$$

Moreover, in this case, the infimum in (1.5) is not attained. For if u_0 is a minimizer for S_V, then $S_V = \|u_0\|^p / |u_0|_{p_s^*}^p > [u_0]_{s,p}^p / |u_0|_{p_s^*}^p \geq S_{s,p}$, a contradiction. It is expected that the minimizers φ of $S_{s,p}$ decay at infinity as $\varphi(x) \sim |x|^{-(N-sp)/(p-1)}$ (this occurs for $s = 1$, see [34]) so that, for

$$\alpha_0 = \begin{cases} 1 & \text{if } N > sp^2 \\ \frac{N}{p'(N-sp)} & \text{if } sp < N \leq sp^2, \end{cases}$$

the conclusion $S_V = S_{s,p}$ is expected to hold by assuming that $V \in L^{\alpha'}(\mathbb{R}^N)$ for some $\alpha > \alpha_0$. Since X is compactly embedded in $L_K^p(\mathbb{R}^N)$ by Proposition 1.1,

$$\lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^p}{|u|_{p,K}^p} > 0 \tag{1.6}$$

is the first eigenvalue of the eigenvalue problem

$$(-\Delta)_p^s u + V(x) |u|^{p-2} u = \lambda K(x) |u|^{p-2} u, \quad u \in X. \tag{1.7}$$

First we obtain a positive solution of equation (1.4) when $\lambda < \lambda_1$.

Theorem 1.3. *Assume* (H_1) *and* (H_2). *If* $\lambda < \lambda_1$, *then there exists* $\mu^*(\lambda) > 0$ *such that equation* (1.4) *has a positive solution for all* $\mu \geq \mu^*(\lambda)$.

Let $u^\pm(x) = \max\{\pm u(x), 0\}$ be the positive and negative parts of u, respectively, and set

$$\Phi^+(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} \left(\frac{\lambda}{p} K(x) (u^+)^p + \frac{\mu}{q} L(x) (u^+)^q + \frac{1}{p_s^*} (u^+)^{p_s^*}\right) dx, \quad u \in X.$$

If u is a critical point of Φ^+, then recalling the elementary inequality

$$|a^- - b^-|^p \leq |a - b|^{p-2}(a - b)(b^- - a^-), \quad \forall a, b \in \mathbb{R},$$

we obtain

$$0 = \Phi^{+\prime}(u)(-u^-) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u^-(y) - u^-(x))}{|x - y|^{N+sp}} \, dx dy$$
$$+ \int_{\mathbb{R}^N} V(x)|u^-|^p dx \geq \|u^-\|^p$$

and hence $u^- = 0$, so $u = u^+ \geq 0$ is a critical point of Φ and therefore a nonnegative solution of equation (1.4). Moreover, if it was $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, then (1.4) and (1.1) give

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x_0)} \frac{u(y)^{p-1}}{|x_0 - y|^{N+sp}} \, dy = 0,$$

so $u = 0$. Thus, nontrivial critical points of Φ^+ are positive solutions of (1.4). The proof of Theorem 1.3 will be based on constructing a minimax level of mountain pass type for Φ^+ below the threshold level given in Proposition 1.2.

Next we obtain a (possibly nodal) nontrivial solution of equation (1.4) when $\lambda \geq \lambda_1$.

Theorem 1.4. *Assume* (H_1) *and* (H_2). *If* $\lambda \geq \lambda_1$, *then there exists* $\mu_*(\lambda) > 0$ *such that equation* (1.4) *has a nontrivial solution for all* $\mu \geq \mu_*(\lambda)$.

This extension of Theorem 1.3 is nontrivial. Indeed, the functional Φ does not have the mountain pass geometry when $\lambda \geq \lambda_1$ since the origin is no longer a local minimizer, and a linking type argument is needed. However, the classical linking theorem cannot be used since the nonlinear eigenvalue problem (1.7) does not have linear eigenspaces.

2 Tools from critical point theory

We will use a general construction based on sublevel sets as in Perera and Szulkin [29] (see also Perera et al. [26, Proposition 3.23]). Moreover, the standard sequence of eigenvalues of (1.7) based on the genus does not give enough information about the structure of the sublevel sets to carry out this linking construction. Therefore we will use a different sequence of eigenvalues as in Perera [25] that is based on a cohomological index.

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [13] is defined as follows. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the

classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\}, & A \neq \emptyset \\ 0, & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$. The following proposition summarizes the basic properties of this index.

Proposition 2.1 (Fadell-Rabinowitz [13]). *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

(i_1) *Definiteness: $i(A) = 0$ if and only if $A = \emptyset$;*

(i_2) *Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism;*

(i_3) *Dimension: $i(A) \leq \dim W$;*

(i_4) *Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighborhood $N_\delta(A) = \{u \in W : dist\,(u, A) \leq \delta\}$;*

(i_5) *Subadditivity: If A and B are closed, then $i(A \cup B) \leq i(A) + i(B)$;*

(i_6) *Stability: If SA is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$;*

(i_7) *Piercing property: If A, A_0 and A_1 are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$;*

(i_8) *Neighborhood of zero: If U is a bounded closed symmetric neighborhood of 0, then $i(\partial U) = \dim W$.*

Eigenvalues of problem (1.7) coincide with critical values of the functional

$$\Psi(u) = \left(\int_{\mathbb{R}^N} K(x)\,|u|^p\,dx\right)^{-1}$$

on the unit sphere $S = \{u \in X : \|u\| = 1\}$, and we can define an increasing and unbounded sequence of eigenvalues via a suitable minimax scheme. The standard scheme based on the genus does not give the index information necessary to prove Theorem 1.4, so we will use a different scheme based on a cohomological index as in Perera [25]. First we note that the general theory

developed in Perera et al. [26] applies to this problem. Indeed, the odd $(p-1)$-homogeneous operator $A_p \in C(X, X^*)$, where X^* is the dual of X, defined by

$$A_p(u) v = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} \, dx dy$$

$$+ \int_{\mathbb{R}^N} V(x) |u|^{p-2} uv \, dx, \quad u, v \in X \quad (2.1)$$

that is associated with the left-hand side of equation (1.7) is the Fréchet derivative of the C^1-functional $X \to \mathbb{R}$, $u \mapsto \|u\|^p / p$ and satisfies for all $u, v \in X$,

$$A_p(u) u = \|u\|^p \quad (2.2)$$

and

$$|A_p(u) v| \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-1} |v(x) - v(y)|}{|x - y|^{N+sp}} \, dx dy + \int_{\mathbb{R}^N} V(x) |u|^{p-1} |v| \, dx$$

$$\leq [u]_{s,p}^{p-1} [v]_{s,p} + |u|_{p,V}^{p-1} |v|_{p,V} \leq \|u\|^{p-1} \|v\| \quad (2.3)$$

by the Hölder inequalities for integrals and sums. Moreover, since X is uniformly convex, it follows from (2.2) and (2.3) that A_p is of type (S), i.e., every sequence $(u_j) \subset X$ such that

$$u_j \rightharpoonup u, \quad A_p(u_j) (u_j - u) \to 0$$

has a subsequence that converges strongly to u (cf. [26, Proposition 1.3]). Hence the operator A_p satisfies the structural assumptions of [26, Chapter 1]. On the other hand, the odd $(p-1)$-homogeneous operator $B_p \in C(X, X^*)$ defined by

$$B_p(u) v = \int_{\mathbb{R}^N} K(x) |u|^{p-2} uv \, dx, \quad u, v \in X$$

that appears in the right-hand side of (1.7) is the Fréchet derivative of the C^1-functional $X \to \mathbb{R}$, $u \mapsto |u|_{p,K}^p / p$ and satisfies $B_p(u) u = |u|_{p,K}^p$ for all $u \in X$. Moreover, since X is compactly embedded in $L_K^p(\mathbb{R}^N)$ by Proposition 1.1, B_p is compact. Hence B_p satisfies the assumptions of [26, Chapter 4]. Let $\mathcal{F}$ denote the class of symmetric subsets of S and set

$$\lambda_k := \inf_{M \in \mathcal{F}, \, i(M) \geq k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

By [26, Theorem 4.6], $\lambda_k \nearrow +\infty$ is a sequence of eigenvalues of problem (1.7) and

$$\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(S \setminus \Psi_{\lambda_{k+1}}) = k, \quad (2.4)$$

where $\Psi^{\lambda_k} = \{u \in S : \Psi(u) \leq \lambda_k\}$ and $\Psi_{\lambda_{k+1}} = \{u \in S : \Psi(u) \geq \lambda_{k+1}\}$.
The proof of Theorem 1.4 will make essential use of (2.4) and will be based on the following abstract critical point theorem.

Theorem 2.2. *Let X be a Banach space and let $S = \{u \in X : \|u\| = 1\}$ be the unit sphere in X. Let Φ be a C^1-functional on X and let A_0, B_0 be disjoint nonempty closed symmetric subsets of S such that*

$$i(A_0) = i(S \setminus B_0) < \infty. \tag{2.5}$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup \Phi(A) \leq \inf \Phi(B), \qquad \sup \Phi(D) < \infty,$$

where

$$A = \{tu : u \in A_0,\, 0 \leq t \leq R\} \cup \{R\,\pi((1-t)\,u + tv) : u \in A_0,\, 0 \leq t \leq 1\},$$

$$B = \{ru : u \in B_0\},$$

$$D = \{tu : u \in A,\, \|u\| = R,\, 0 \leq t \leq 1\},$$

and $\pi : X \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ is the radial projection onto S. Let

$$\Gamma = \{\gamma \in C(D, X) : \gamma(D) \text{ is closed and } \gamma|_A = id_A\}$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(D)} \Phi(u).$$

Then $\inf \Phi(B) \leq c \leq \sup \Phi(D)$ and Φ has a $(PS)_c$ sequence.

Theorem 2.2, which does not require a direct sum decomposition, generalizes the linking theorem of Rabinowitz [30] and is proved in Candito et al. [8] (see also Yang and Perera [37]). The linking construction in its proof was also used in Perera and Szulkin [29] to obtain nontrivial solutions of p-Laplacian problems with nonlinearities that interact with the spectrum. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [11]. See also Perera et al. [26, Proposition 3.23].

3   Preliminaries

In this preliminary section we prove Propositions 1.1 and 1.2.

Proof of Proposition 1.1. Let (u_j) be a bounded sequence in X. Then, a renamed subsequence converges weakly and a.e. to some $u \in X$. By virtue of assumption (H_2), given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$K(x) \leq \varepsilon\, V(x), \qquad L(x) \leq \varepsilon\, V(x)^{[Np-(N-sp)q]/sp^2}, \qquad \text{for all } x \in B_{r_\varepsilon}(0)^c.$$

The first inequality gives

$$\int_{B_{r_\varepsilon}(0)^c} K(x)\, |u_j|^p\, dx \leq \varepsilon \int_{\mathbb{R}^N} V(x)\, |u_j|^p\, dx = \varepsilon\, \mathcal{O}(1),$$

$$\int_{B_{r_\varepsilon}(0)^c} K(x)\, |u|^p\, dx \leq \varepsilon \int_{\mathbb{R}^N} V(x)\, |u|^p\, dx. \tag{3.1}$$

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Combining the second inequality with the Young's inequality gives

$$L(x) |t|^q \le \varepsilon\big(V(x) |t|^p + |t|^{p_s^*}\big), \qquad \text{for all } x \in B_{r_\varepsilon}(0)^c \text{ and } t \in \mathbb{R},$$

so that

$$\int_{B_{r_\varepsilon}(0)^c} L(x) |u_j|^q \, dx \le \varepsilon \int_{\mathbb{R}^N} \Big(V(x) |u_j|^p + |u_j|^{p_s^*}\Big) \, dx = \varepsilon \, \mathcal{O}(1),$$

$$\int_{B_{r_\varepsilon}(0)^c} L(x) |u|^q \, dx \le \varepsilon \int_{\mathbb{R}^N} \Big(V(x) |u|^p + |u|^{p_s^*}\Big) \, dx. \quad (3.2)$$

Since X is compactly embedded in $L^p(B_{r_\varepsilon}(0)) \cap L^q(B_{r_\varepsilon}(0))$ and K and L are bounded, $u_j \to u$ in $L^p_K(B_{r_\varepsilon}(0)) \cap L^q_L(B_{r_\varepsilon}(0))$ for a further subsequence. Then it follows from (3.1) and (3.2) that $u_j \to u$ in $L^p_K(\mathbb{R}^N) \cap L^q_L(\mathbb{R}^N)$, since they are uniformly convex spaces. $\qquad\square$

Proof of Proposition 1.2. Let (u_j) be a $(\mathrm{PS})_c$ sequence. First we show that (u_j) is bounded in X. We have

$$\Phi(u_j) = \frac{1}{p} \, \|u_j\|^p - \int_{\mathbb{R}^N} \left(\frac{\lambda}{p} \, K(x) |u_j|^p + \frac{\mu}{q} \, L(x) |u_j|^q + \frac{1}{p_s^*} |u_j|^{p_s^*} \right) dx = c + \mathrm{o}(1) \qquad (3.3)$$

and

$$\Phi'(u_j) \, v = \int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2} \, (u_j(x) - u_j(y)) \, (v(x) - v(y))}{|x - y|^{N+sp}} \, dxdy$$

$$+ \int_{\mathbb{R}^N} V(x) |u_j|^{p-2} \, u_j \, v \, dx - \int_{\mathbb{R}^N} \Big(\lambda K(x) |u_j|^{p-2} \, u_j + \mu L(x) |u_j|^{q-2} \, u_j + |u_j|^{p_s^*-2} \, u_j \Big) v \, dx$$

$$= \mathrm{o}(\|v\|) \quad \forall v \in X, \quad (3.4)$$

in particular,

$$\Phi'(u_j) \, u_j = \|u_j\|^p - \int_{\mathbb{R}^N} \Big(\lambda K(x) |u_j|^p + \mu L(x) |u_j|^q + |u_j|^{p_s^*} \Big) \, dx = \mathrm{o}(\|u_j\|). \qquad (3.5)$$

By (3.3) and (3.5),

$$\mu \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} L(x) |u_j|^q \, dx + \left(\frac{1}{p} - \frac{1}{p_s^*} \right) \int_{\mathbb{R}^N} |u_j|^{p_s^*} \, dx = \mathrm{o}(\|u_j\|) + \mathcal{O}(1),$$

which gives

$$\int_{\mathbb{R}^N} L(x) |u_j|^q \, dx \le \mathrm{o}(\|u_j\|) + \mathcal{O}(1), \qquad \int_{\mathbb{R}^N} |u_j|^{p_s^*} \, dx \le \mathrm{o}(\|u_j\|) + \mathcal{O}(1) \qquad (3.6)$$

since $\mu > 0$ and $p < q < p_s^*$. By (H_2), there exists $r > 0$ such that $\lambda K(x) \le V(x)/2$ for all $x \in B_r(0)^c$ and hence

$$\int_{B_r(0)^c} \lambda K(x) |u_j|^p \, dx \le \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u_j|^p \, dx \le \frac{1}{2} \, \|u_j\|^p, \qquad (3.7)$$

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and by the Hölder inequality,

$$\int_{B_r(0)} \lambda K(x) |u_j|^p \, dx \leq |\lambda| \, |K|_\infty \operatorname{vol}(B_r(0))^{sp/N} \left(\int_{\mathbb{R}^N} |u_j|^{p_s^*} \, dx \right)^{p/p_s^*}, \tag{3.8}$$

where $|\cdot|_\infty$ is the norm in $L^\infty(\mathbb{R}^N)$. Since $p > 1$, it follows from (3.5)–(3.8) that (u_j) is bounded. So a renamed subsequence of (u_j) converges to some u weakly in X, strongly in $L_K^p(\mathbb{R}^N) \cap L_L^q(\mathbb{R}^N)$ by Proposition 1.1, and a.e. in $\mathbb{R}^N$. The sequence $|u_j(x) - u_j(y)|^{p-2} (u_j(x) - u_j(y))/|x-y|^{(N+sp)/p'}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ and it converges to $|u(x) - u(y)|^{p-2} (u(x) - u(y))/|x-y|^{(N+sp)/p'}$ almost everywhere in $\mathbb{R}^{2N}$. Moreover, $(v(x) - v(y))/|x-y|^{(N+sp)/p} \in L^p(\mathbb{R}^{2N})$, so the first integral in formula (3.4) converges to

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x-y|^{N+sp}} \, dx \, dy$$

for a subsequence. Similarly,

$$\int_{\mathbb{R}^N} V(x) |u_j|^{p-2} u_j v \, dx \to \int_{\mathbb{R}^N} V(x) |u|^{p-2} uv \, dx,$$

$$\int_{\mathbb{R}^N} K(x) |u_j|^{p-2} u_j v \, dx \to \int_{\mathbb{R}^N} K(x) |u|^{p-2} uv \, dx,$$

$$\int_{\mathbb{R}^N} L(x) |u_j|^{q-2} u_j v \, dx \to \int_{\mathbb{R}^N} L(x) |u|^{q-2} uv \, dx,$$

$$\int_{\mathbb{R}^N} |u_j|^{p_s^*-2} u_j v \, dx \to \int_{\mathbb{R}^N} |u|^{p_s^*-2} uv \, dx$$

for a further subsequence. So passing to the limit in (3.4) shows that u is a weak solution of equation (1.4). Suppose now that $u = 0$. Since (u_j) is bounded in X and it converges to 0 in $L_K^p(\mathbb{R}^N) \cap L_L^q(\mathbb{R}^N)$, (3.5) and (1.5) give

$$\mathrm{o}(1) = \|u_j\|^p - \int_{\mathbb{R}^N} |u_j|^{p_s^*} \, dx \geq \|u_j\|^p \left(1 - \frac{\|u_j\|^{p_s^*-p}}{S_V^{p_s^*/p}} \right).$$

If $\|u_j\| \to 0$, then $\Phi(u_j) \to 0$, contradicting $c \neq 0$, so this implies

$$\|u_j\|^p \geq S_V^{N/sp} + \mathrm{o}(1)$$

for a further subsequence. Then (3.3) and (3.5) give

$$c = \left(\frac{1}{p} - \frac{1}{p_s^*} \right) \|u_j\|^p + \mathrm{o}(1) \geq \frac{s}{N} S_V^{N/sp} + \mathrm{o}(1),$$

contradicting $c < \dfrac{s}{N} S_V^{N/sp}$. This concludes the proof. $\qquad\square$

4 Proofs

In this section we prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Fix $u_0 > 0$ in X such that $|u_0|_{p_s^*} = 1$. Since $p < q < p_s^*$,

$$\Phi^+(tu_0) = \frac{t^p}{p} \, \|u_0\|^p - \frac{\lambda \, t^p}{p} \, |u_0|_{p,K}^p - \frac{\mu \, t^q}{q} \, |u_0|_{q,L}^q - \frac{t^{p_s^*}}{p_s^*} \to -\infty$$

as $t \to +\infty$. Take $t_0 > 0$ so large that $\Phi^+(t_0 u_0) \leq 0$, let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \, \gamma(1) = t_0 u_0\}$$

be the class of paths joining 0 and $t_0 u_0$, and set

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \Phi^+(u).$$

By Proposition 1.1, we learn that

$$T = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^p}{|u|_{q,L}^p} > 0. \tag{4.1}$$

By formulas (1.6), (4.1), and (1.5), we obtain

$$\Phi^+(u) \geq \frac{1}{p} \left(1 - \frac{\lambda^+}{\lambda_1}\right) \|u\|^p - \frac{\mu}{q} \, T^{-q/p} \, \|u\|^q - \frac{1}{p_s^*} \, S_V^{-p_s^*/p} \, \|u\|^{p_s^*} \quad \forall u \in X,$$

where $\lambda^+ = \max\{\lambda, 0\}$. Since $\lambda^+ < \lambda_1$ and $p_s^* > q > p$, it follows from this that 0 is a strict local minimizer of Φ^+, so $c > 0$. Thus, Φ^+ has a $(\mathrm{PS})_c$ sequence (u_j) by the Mountain Pass Theorem. Since $\gamma(s) = s t_0 u_0$ is a path in Γ,

$$c \leq \max_{s \in [0,1]} \Phi^+(s t_0 u_0) \leq \max_{t \geq 0} \Phi^+(t u_0) \leq \max_{t \geq 0} \left[\frac{t^p}{p} \left(\|u_0\|^p - \lambda \, |u_0|_{p,K}^p\right) - \frac{\mu \, t^q}{q} \, |u_0|_{q,L}^q\right]$$

$$= \left(\frac{1}{p} - \frac{1}{q}\right) \frac{\left(\|u_0\|^p - \lambda \, |u_0|_{p,K}^p\right)^{q/(q-p)}}{\left(\mu \, |u_0|_{q,L}^q\right)^{p/(q-p)}} < \frac{s}{N} \, S_V^{N/sp}$$

if $\mu > 0$ is sufficiently large. An argument similar to that in the proof of Proposition 1.2 now shows that a subsequence of (u_j) converges weakly to a positive solution of equation (1.4). $\square$

Turning to the proof of Theorem 1.4, since $\lambda \geq \lambda_1$, $\lambda_k \leq \lambda < \lambda_{k+1}$ for some $k \geq 1$. By (2.4), $i(\Psi^{\lambda_k}) = i(S \setminus \Psi_{\lambda_{k+1}}) = k$. First, we construct a compact symmetric subset A_0 of Ψ^{λ_k} with the same index. As we have already noted, the operator A_p defined in (2.1) satisfies the structural assumptions of [26, Chapter 1].

Lemma 4.1. *The operator A_p is strictly monotone, i.e.,*

$$(A_p(u) - A_p(v)) (u - v) > 0$$

for all $u \neq v$ in X.

Proof. It is easily seen from (2.3) that $A_p(u)\,v \leq \|u\|^{p-1}\,\|v\|$ for all $u, v \in X$ and the equality holds if and only if $\alpha u = \beta v$ a.e. in $\mathbb{R}^N$ for some $\alpha, \beta \geq 0$, not both zero, so the conclusion follows from [26, Lemma 6.3]. □

Lemma 4.2. *For each $w \in L_K^p(\mathbb{R}^N)$, the equation*

$$(-\Delta)_p^s\, u + V(x)\,|u|^{p-2}\,u = K(x)\,|w|^{p-2}\,w, \quad x \in \mathbb{R}^N \tag{4.2}$$

admits a unique weak solution $u \in X$. Furthermore, the mapping $J : L_K^p(\mathbb{R}^N) \to X$, $w \mapsto u$ is continuous.

Proof. The existence follows from a standard minimization argument since X is continuously embedded in $L_K^p(\mathbb{R}^N)$ by Proposition 1.1, and the uniqueness is immediate from the strict monotonicity of the operator A_p. Let $w_j \to w$ in $L_K^p(\mathbb{R}^N)$ and let $u_j = J(w_j)$, so

$$A_p(u_j)\,v = \int_{\mathbb{R}^N} K(x)\,|w_j|^{p-2}\,w_j\,v\,dx \quad \forall v \in X. \tag{4.3}$$

Testing with $v = u_j$ gives

$$\|u_j\|^p = \int_{\mathbb{R}^N} K(x)\,|w_j|^{p-2}\,w_j\,u_j\,dx \leq |w_j|_{p,K}^{p-1}\,|u_j|_{p,K}$$

by the Hölder inequality, which together with the continuity of the embedding $X \hookrightarrow L_K^p(\mathbb{R}^N)$ shows that (u_j) is bounded in X. So a renamed subsequence of (u_j) converges to some u weakly in X, strongly in $L_K^p(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$. As in the proof of Proposition 1.2,

$$A_p(u_j)\,v \to A_p(u)\,v, \qquad \int_{\mathbb{R}^N} K(x)\,|w_j|^{p-2}\,w_j\,v\,dx \to \int_{\mathbb{R}^N} K(x)\,|w|^{p-2}\,wv\,dx,$$

along a subsequence. So passing to the limit in (4.3) shows that u is a weak solution of equation (4.2) and hence $u = J(w)$. Testing (4.3) with $u_j - u$ gives

$$A_p(u_j)\,(u_j - u) = \int_{\mathbb{R}^N} K(x)\,|w_j|^{p-2}\,w_j\,(u_j - u)\,dx \leq |w_j|_{p,K}^{p-1}\,|u_j - u|_{p,K} \to 0,$$

so $u_j \to u$ for a further subsequence since A_p is of type (S). □

Proposition 4.3. *If $\lambda_k < \lambda_{k+1}$, then Ψ^{λ_k} has a compact symmetric subset A_0 with $i(A_0) = k$.*

Proof. Let

$$\pi_{p,K}(u) = \frac{u}{|u|_{p,K}}, \quad u \in X \setminus \{0\},$$

be the radial projection onto $S_{p,K} = \{u \in X : |u|_{p,K} = 1\}$, and let

$$A = \pi_{p,K}(\Psi^{\lambda_k}) = \{w \in S_{p,K} : \|w\|^p \leq \lambda_k\}.$$

Then $i(A) = i(\Psi^{\lambda_k}) = k$ by (i_2) of Proposition 2.1 and (2.4). For $w \in A$, let $u = J(w)$, where J is the map defined in Lemma 4.2, so

$$A_p(u)\,v = \int_{\mathbb{R}^N} K(x)\,|w|^{p-2}\,wv\,dx, \quad \forall v \in X.$$

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Testing with $v = u, w$ and using the Hölder inequality gives

$$\|u\|^p \le |w|_{p,K}^{p-1} \, |u|_{p,K} = |u|_{p,K}, \qquad 1 = A_p(u) \, w \le \|u\|^{p-1} \, \|w\|,$$

so

$$\|\pi_{p,K}(u)\| = \frac{\|u\|}{|u|_{p,K}} \le \|w\|$$

and hence $\pi_{p,K}(u) \in A$. Let $\widetilde{J} = \pi_{p,K} \circ J$ and let $\widetilde{A} = \widetilde{J}(A) \subset A$. Since the embedding $X \hookrightarrow L_K^p(\mathbb{R}^N)$ is compact by Proposition 1.1 and $\widetilde{J}$ is an odd continuous map from $L_K^p(\mathbb{R}^N)$ to X, then $\widetilde{A}$ is a compact set and $i(\widetilde{A}) = i(A) = k$. Let

$$\pi(u) = \frac{u}{\|u\|}, \quad u \in X \setminus \{0\} \tag{4.4}$$

be the radial projection onto S and let $A_0 = \pi(\widetilde{A})$. Therefore $A_0 \subset \Psi^{\lambda_k}$ is compact and $i(A_0) = i(\widetilde{A}) = k$. $\qquad\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We apply Theorem 2.2. Since $\lambda \ge \lambda_1$, $\lambda_k \le \lambda < \lambda_{k+1}$ for some $k \ge 1$. Then Ψ^{λ_k} has a compact symmetric subset A_0 with

$$i(A_0) = k$$

by Proposition 4.3. We take $B_0 = \Psi_{\lambda_{k+1}}$, so that

$$i(S \setminus B_0) = k$$

by (2.4). So (2.5) holds. For $u \in S$ and $t \ge 0$,

$$\Phi(tu) = \frac{t^p}{p} \left(1 - \frac{\lambda}{\Psi(u)} \right) - \frac{\mu \, t^q}{q} \, |u|_{q,L}^q - \frac{t^{p_s^*}}{p_s^*} \, |u|_{p_s^*}^{p_s^*}. \tag{4.5}$$

Pick any $v \in S \setminus A_0$. Since A_0 is compact, so is the set

$$X_0 = \{\pi((1-t) \, u + tv) : u \in A_0, \, 0 \le t \le 1\},$$

where π is as in (4.4), and hence

$$\alpha = \sup_{u \in X_0} \left(1 - \frac{\lambda}{\Psi(u)} \right) < \infty, \qquad \beta = \inf_{u \in X_0} |u|_{q,L}^q > 0.$$

For $u \in A_0$, (4.5) gives

$$\Phi(tu) \le -\frac{t^p}{p} \left(\frac{\lambda}{\lambda_k} - 1 \right) \le 0 \tag{4.6}$$

since $\lambda \ge \lambda_k$. For $u \in X_0$, (4.5) gives

$$\Phi(tu) \le \frac{\alpha \, t^p}{p} - \frac{\mu \beta \, t^q}{q} \le \left(\frac{1}{p} - \frac{1}{q} \right) \frac{(\alpha^+)^{q/(q-p)}}{(\mu \beta)^{p/(q-p)}}, \tag{4.7}$$

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where $\alpha^+ = \max\{\alpha, 0\}$. Fix $\mu > 0$ so large that the last expression is less than $(s/N)S_V^{N/sp}$, take positive $R \geq (q\,\alpha^+/p\,\mu\beta)^{1/(q-p)}$, and let A and D be as in Theorem 2.2. Then it follows from (4.6) and (4.7) that

$$\sup \Phi(A) \leq 0, \qquad \sup \Phi(D) < \frac{s}{N}\, S_V^{N/sp}.$$

Finally for $u \in B_0$, (4.5) gives

$$\Phi(tu) \geq \frac{t^p}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{\mu\, t^q}{q}\, T^{-q/p} - \frac{t^{p_s^*}}{p_s^*}\, S_V^{-p_s^*/p},$$

where T is as in (4.1). Since $\lambda < \lambda_{k+1}$ and $p_s^* > q > p$, it follows from this that if $0 < r < R$ is sufficiently small and B is as in Theorem 2.2, then

$$\inf \Phi(B) > 0.$$

Thus, $0 < c < (s/N)S_V^{N/sp}$ and Φ has a $(PS)_c$ sequence by Theorem 2.2, a subsequence of which converges weakly to a nontrivial solution of equation (1.4) by Proposition 1.2. $\qquad\square$

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